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# Synchrony Patterns in Coupled Network Dynamics

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In many applications, ODEs are associated with a given network, and the form of these equations reflects the network architecture. For example in a neuroscience model, nodes might represent neurons and edges axons, coupled via electrical signals passing along the axons. The state of each node  $i$  is represented by a variable  $x_i$ , which might be a scalar or a vector.

Each node typically has an internal dynamic, an ODE that determines how it would behave if it were not coupled to other nodes. Connections from one node to another lead to coupling terms in the equations: if there is an input from node  $j$  to node  $i$ , then  $\frac{dx_i}{dt}$  is a function of both  $x_i$  and  $x_j$ .



### Example ( $\beta$ IG model)

The  $\beta$ IG model of diabetes, Topp et al. (2000), takes the form

$$\begin{aligned}\dot{G} &= a - (b + cI)G \\ \dot{I} &= \beta \left( \frac{dG^2}{e + G^2} \right) - fI \\ \dot{\beta} &= (-g + hG - iG^2)/\beta\end{aligned}\tag{1}$$

Here dots are time derivatives. The terms  $G$  = glucose level,  $I$  = insulin level, and  $\beta$  = beta-cell mass depend on time  $t$ . The other terms  $a, b, c, d, e, f, g, h, i$  are parameters, whose value is constant during any particular run of the model or in any particular real system.

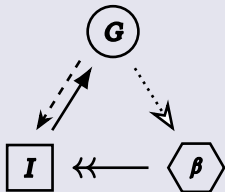


### Example ( $\beta$ IG model)

The network structure arises when we consider which variables depend on which.

Here:

- ▶ The change in  $G$  depends on  $G, I$  but not on  $\beta$ .
- ▶ The change in  $I$  depends on  $G, I$ , and  $\beta$ .
- ▶ The change in  $\beta$  depends on  $G, \beta$  but not on  $I$ .



It is natural to encode these relationships as the network. Here each variable is represented by a cell symbol (circle, square, hexagon) and arrows show which variable affects any given cell variable. The different cell symbols indicate different 'cell types', meaning that the form of the equation is different for those cells.



### Example (FitzHugh-Nagumo neurons)

Consider an ODE system representing three coupled FitzHugh-Nagumo neurons:

$$\begin{aligned} \dot{v}_1 &= v_1(a - v_1)(v_1 - 1) - w_1 - cv_2 & \dot{w}_1 &= bv_1 - \gamma w_1 \\ \dot{v}_2 &= v_2(a - v_2)(v_2 - 1) - w_2 - cv_3 & \dot{w}_2 &= bv_2 - \gamma w_2 \\ \dot{v}_3 &= v_3(a - v_3)(v_3 - 1) - w_3 - cv_1 & \dot{w}_3 &= bv_3 - \gamma w_3 \end{aligned} \quad (2)$$

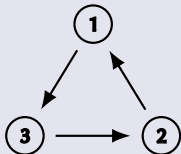
Here  $v_i$  is the membrane potential of cell  $i$ ,  $w_i$  is a surrogate for an ionic current, and  $a$ ;  $b$ ;  $\gamma$  are parameters with  $0 < a < 1$ ,  $b > 0$ ;  $\gamma > 0$ .

In (2) the dynamic equations are the same for each neuron, subject to appropriate permutations of the variables.



### Example (FitzHugh-Nagumo neurons)

In other words, the individual neurons are identical, and the couplings are also identical.



$$\dot{x}_1 = f(x_1, x_2),$$

$$\dot{x}_2 = f(x_2, x_3),$$

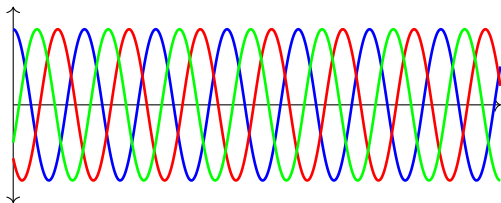
$$\dot{x}_3 = f(x_3, x_1),$$

The state space of cell  $i$  is now 2-dimensional, with variables  $(v_i, w_i)$ . Because the variables enter the equations in the same manner for each  $i$ , subject to the cyclic permutation  $\mathbb{Z}_3$ , the cells have the same type and so do the arrows. In the diagram, we represent this by using circles for all three cells and the same kind of arrow for all three couplings.

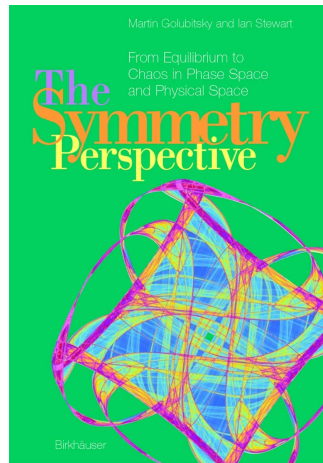


We point out that when  $a = b = \gamma = 0.5$  and  $c = 2$ , system (3) has a stable periodic state in which successive cells are one-third of a period out of phase. Below we show the pattern for  $v_j$ ; the same pattern occurs for  $w_j$ . This state is a *discrete rotating wave* that exhibits spatiotemporal symmetry caused by the action of  $\mathbb{Z}_3$ :

$$x_2(t) = x_1(t - T/3) \quad x_3(t) = x_1(t - 2T/3)$$



**This kind of dynamics (with the action of a symmetry group) is called Equivariant Dynamics!**





With appropriate adjustments, many results of equivariant dynamics apply to symmetric networks. However, few models in applied science exhibit global symmetries.

In 2002, Marcus Pivato described a 16-cell network that had a periodic state in which the nodes were partitioned into four subsets of four nodes. [14]

The cells in each partition were synchronous, while cells in different partitions exhibited the same dynamics within a phase shift multiple of  $1/4$  of the period. This was a rotating wave induced by  $\mathbb{Z}_4$ , except that the network **lacks**  $\mathbb{Z}_4$  symmetry.



Around 2003, Marty Golubitsky, Ian Stewart, and other collaborators proposed a theory to study this type of network based on the notion of local symmetry, establishing the **groupoid formalism**. [9, 11]

Mathematically, a network is represented by a directed graph whose nodes and edges are classified according to associated labels or “types”. The nodes (or “cells”) of a (directed and labeled) network  $\mathcal{N}$  represent dynamical systems (state variables), and the edges (“arrows”) represent couplings, interactions between these variables.

Formally, we have:

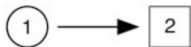


## Coupled Network Definition

A coupled cell network  $\mathcal{N} = \mathcal{N}(\mathcal{C}, \mathcal{E})$  satisfies the following conditions:

1. There is a finite set  $\mathcal{C} = \{1, 2, \dots, n\}$  of cells;
2. There is a finite set  $\mathcal{E}$  of arrows.
3. Each arrow  $e$  has a head cell  $\mathcal{H}(e) \in \mathcal{C}$  and a tail cell  $\mathcal{T}(e) \in \mathcal{C}$ .
4. Cells are classified into types. Formally, this is done by defining an equivalence relation  $\sim_{\mathcal{C}}$  on  $\mathcal{C}$ , called cell equivalence. Cells are equivalent if they have the same type.
5. Arrows are also classified into types by defining an equivalence relation  $\sim_{\mathcal{E}}$  on  $\mathcal{E}$ , called arrow equivalence. Arrows are equivalent if they have the same type.
6. Types satisfy two compatibility conditions. If  $e_1, e_2 \in \mathcal{E}$  are  $\sim_{\mathcal{E}}$ -equivalent, then  $\mathcal{H}(e_1)$  e  $\mathcal{H}(e_2)$  are  $\sim_{\mathcal{C}}$ -equivalent, and similarly,  $\mathcal{T}(e_1)$  e  $\mathcal{T}(e_2)$ .

# Coupled Network Dynamics



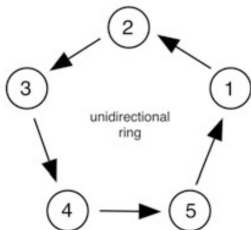
unidirectional coupling



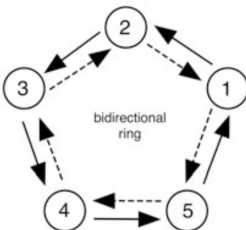
bidirectional coupling



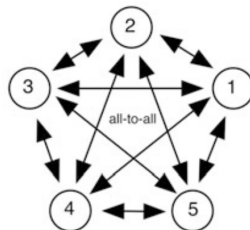
symmetric coupling



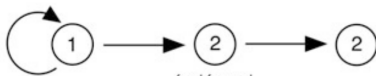
unidirectional ring



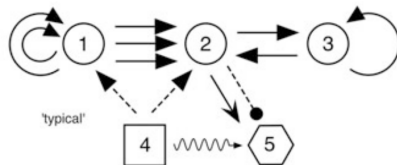
bidirectional ring



all-to-all



feed-forward



'typical'



## Definition of Input Set

Let  $c \in C$ . The *input set* of  $c$  is the set  $I(c)$  of all arrows  $e \in \mathcal{E}$  such that  $\mathcal{H}(e) = c$ . That is,

$$I(c) = \{e \in \mathcal{E} \mid \mathcal{H}(e) = c\}$$

## Definition of Input Isomorphism

An *input isomorphism*  $\beta : I(c) \rightarrow I(d)$  is a bijection between input sets that preserves the arrow type, that is,  $e \sim_E \beta(e)$  for all  $\beta$  and every arrow  $e \in I(c)$ .

(Note that  $\beta^{-1}(e')$  is  $\sim_E$ -equivalent to  $e'$  for all  $e' \in I(d)$ )



If there exists an input isomorphism  $\beta : I(c) \rightarrow I(d)$ , we say that  $c, d$  are *input-isomorphic* or **input-equivalent**.

## Definition of $\sim_E$ -equivalence

The input-equivalence relation  $\sim_E$  in  $\mathcal{C}$  is defined by  $c \sim_E d$  if and only if there exists a bijection

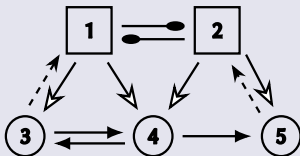
$$\beta : I(c) \rightarrow I(d)$$

such that, for each  $i \in I(c)$ ,

$$i \sim_E \beta(i)$$

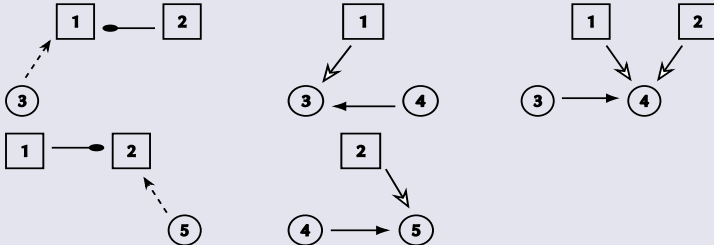
## Example (input isomorphism)

Let  $\mathcal{N}$  be the network shown below:



We see that cells 1 and 2 are input isomorphic, as are cells 3 and 5. However, cells 1 and 3 are not input isomorphic. Although they both receive two inputs, the arrow types are different.

## Example (input sets)



From left to right:  $I(1)$ ,  $I(3)$ ,  $I(4)$ ,  $I(2)$ ,  $I(5)$ .

Strictly, the arrows constitute the input set, but is convenient to show the head and tail cells as well.



We will now see how synchrony manifests itself in networks.

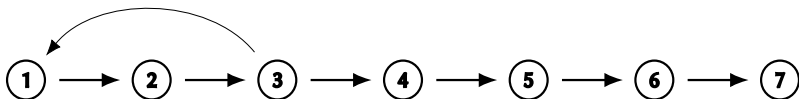
A **polydiagonal** is a subspace

$$\Delta = \{x \mid x_c = x_d \text{ for some subset of cells}\}$$

A **synchrony subspace** is a flow-invariant polydiagonal.



Consider the following feed-forward network with its admissible functions:



$$\dot{x}_1 = f(x_1, x_3),$$

$$\dot{x}_2 = f(x_2, x_1),$$

$$\dot{x}_3 = f(x_3, x_2),$$

$$\dot{x}_4 = f(x_4, x_3),$$

$$\dot{x}_5 = f(x_5, x_4),$$

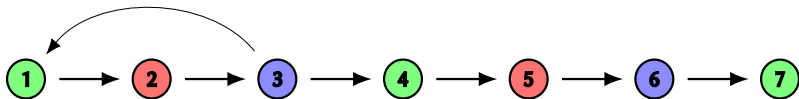
$$\dot{x}_7 = f(x_7, x_6),$$

$$\dot{x}_7 = f(x_7, x_6),$$

We can split this network using **colorings** where

$\Delta = \{x \mid x_1 = x_4 = x_7; x_2 = x_5; x_3 = x_6\}$  is flow-invariant.

Consider the following feed-forward network with its admissible functions:



$$\dot{x}_1 = f(x_1, x_3),$$

$$\dot{x}_2 = f(x_2, x_1),$$

$$\dot{x}_3 = f(x_3, x_2),$$

$$\dot{x}_4 = f(x_4, x_3),$$

$$\dot{x}_5 = f(x_5, x_4),$$

$$\dot{x}_7 = f(x_7, x_6),$$

$$\dot{x}_7 = f(x_7, x_6),$$

We can split this network using **colorings** where

$\Delta = \{x \mid x_1 = x_4 = x_7; x_2 = x_5; x_3 = x_6\}$  is flow-invariant.



## Definition (Coloring)

A *coloring* of a network  $\mathcal{G}$  is a map

$$\kappa : C \rightarrow K$$

where  $K$  is a set whose elements are called *colors*.

We say that cells  $c$  and  $d$  *have the same color* if  $\kappa(c) = \kappa(d)$  and we write  $c \sim_{\kappa} d$  (color equivalence).

A coloring  $\kappa$  is *balanced* if, whenever  $c$  and  $d$  have the same color, then there exists an input isomorphism  $\beta : I(c) \rightarrow I(d)$  such that  $i$  and  $\beta(i)$  have the same color for all  $i \in \mathcal{T}(I(i))$ .



In practice, a coloring is balanced if there exists an input isomorphism that preserves the colors of any two cells of the same color. Since cells of the same color must be input equivalent, a balanced coloring is a refinement of input equivalence.

The *polydiagonal* defined by a coloring  $\kappa$  of  $\mathcal{G}$  is the space

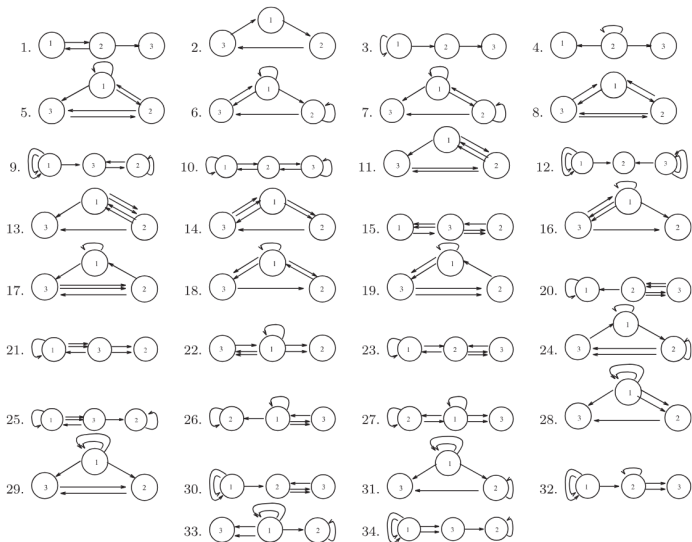
$$\Delta_{\kappa} = \{x \mid \kappa(c) = \kappa(d) \implies x_c = x_d\}$$

That is, cells of the same color are synchronous for  $x \in \Delta_{\kappa}$ .

## Theorem

A polydiagonal  $\Delta_{\kappa}$  is invariant for every admissible function if and only if  $\kappa$  is balanced.

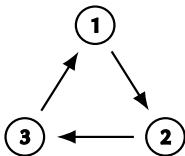
# A catalog of Coupled Network Dynamics



## Reproduced from M. Leite, M. Golubitsky. **Homogeneous three-cell networks. Nonlinearity 19 (2006) 2313–2363**

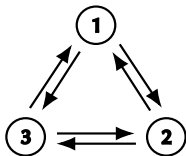
$A_{\#}$	E'vals	Eigenvectors	$A_{\#}$	E'vals	Eigenvectors
$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	1 0 -1	(1, 1, 1) (0, 0, 1) (1, -1, 1)	$A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	1 $\frac{-1 - \sqrt{3}i}{2}$ $\frac{-1 + \sqrt{3}i}{2}$	(1, 1, 1) $\left(1, -\frac{1 + \sqrt{3}i}{2}, -\frac{1 - \sqrt{3}i}{2}\right)$ $\left(1, -\frac{1 - \sqrt{3}i}{2}, -\frac{1 + \sqrt{3}i}{2}\right)$
$A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	1 $0^2$	(1, 1, 1) (0, 0, 1)	$A_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	1 0 0	(1, 1, 1) (1, 0, 0) (0, 0, 1)
$A_5 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	2 0 -1	(1, 1, 1) (-1, 1, 1) (1, -2, 1)	$A_6 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	2 $0^2$	(1, 1, 1) (1, -1, -1)
$A_7 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	2 0 0	(1, 1, 1) (1, -1, 0) (0, 0, 1)	$A_8 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	2 -1 -1	(1, 1, 1) (1, -1, 0) (0, 1, -1)
$A_9 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	2 $\frac{1}{2} + \frac{\sqrt{5}}{2}$ $\frac{1}{2} - \frac{\sqrt{5}}{2}$	(1, 1, 1) (0, 2, -1 + $\sqrt{5}$ ) (0, 2, -1 - $\sqrt{5}$ )	$A_{10} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	2 1 -1	(1, 1, 1) (1, -2, 1) (1, 0, -1)

$$\begin{aligned}\dot{x}_1 &= f(x_1, x_3), \\ \dot{x}_2 &= f(x_2, x_1), \\ \dot{x}_3 &= f(x_3, x_2),\end{aligned}$$



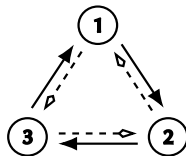
$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned}\dot{x}_1 &= g(x_1, \overline{x_2, x_2}), \\ \dot{x}_2 &= g(x_2, \overline{x_3, x_1}), \\ \dot{x}_3 &= g(x_3, \overline{x_1, x_2}),\end{aligned}$$

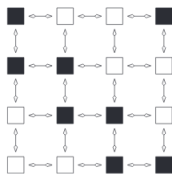


$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

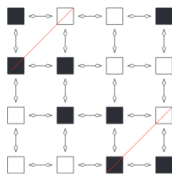
$$\begin{aligned}\dot{x}_1 &= h(x_1, x_2, x_2), \\ \dot{x}_2 &= h(x_2, x_3, x_1), \\ \dot{x}_3 &= h(x_3, x_1, x_2).\end{aligned}$$



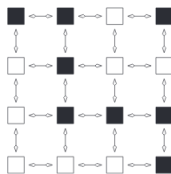
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$



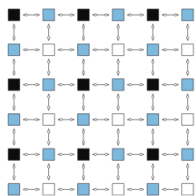
(a)



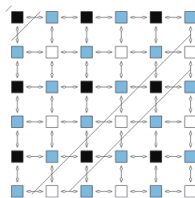
(b)



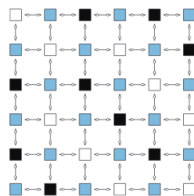
(c)



(a)

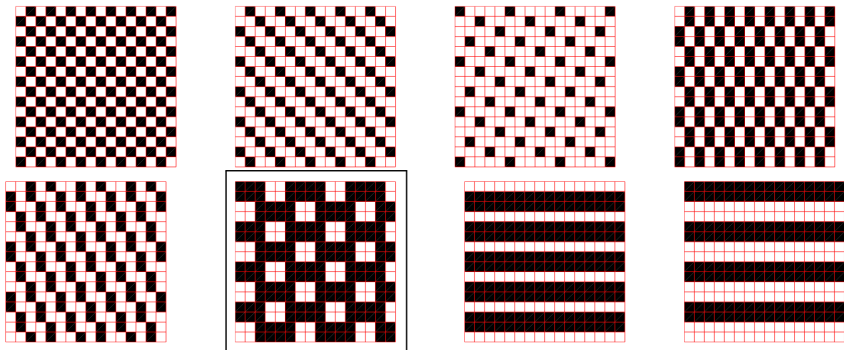


(b)



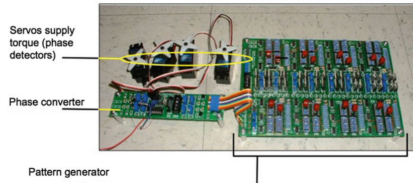
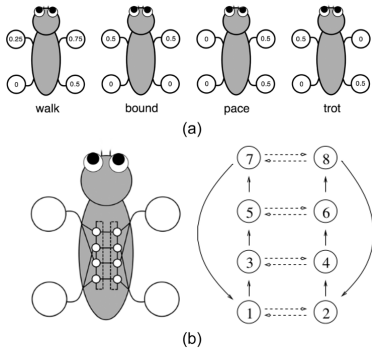
(c)

Wang and Golubitsky [16] list several patterns for two colors.



# Applications

## Central Pattern Generator - CPG



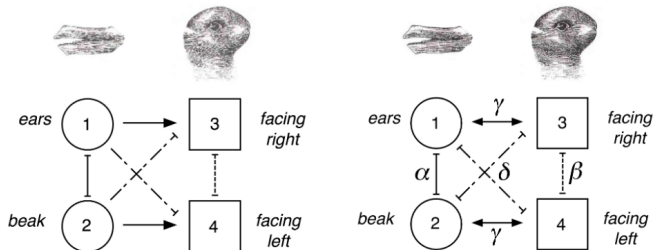
**Fig. 6.1** ANIBOT. To the best of our knowledge, this is the first quadruped robot controlled by a CPG electronic circuit built based on the 8-network proposed by Golubitsky et al. [61, 62, 146, 148]



Article

## Symmetric Networks with Geometric Constraints as Models of Visual Illusions

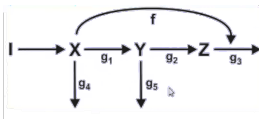
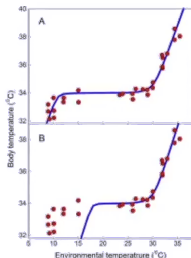
Ian Stewart<sup>1,\*</sup> and Martin Golubitsky<sup>2,t</sup>



**Figure 6.** Network for the rabbit/duck illusion. (Left) One-way connections for geometric constraints. (Right) Two-way connections. Lines ending in bars indicate inhibitory connections, arrows excitatory ones.

Nijhout and Reed (*Integrative Comparative Biology* 54, 2014)

Introduce **chair**  
**escape from homeostasis**  
**input-output map**

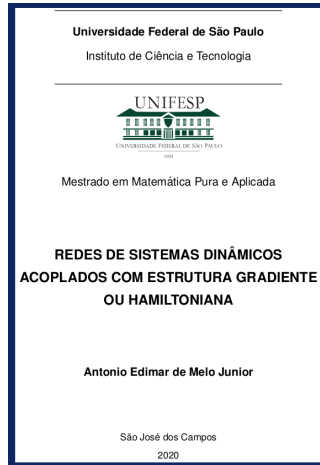
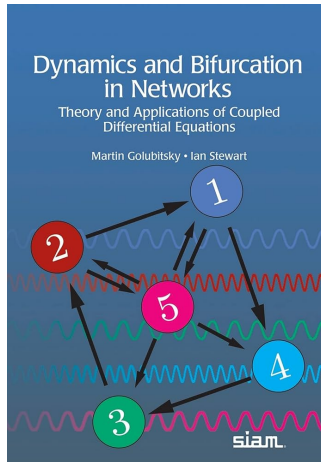


$$\begin{aligned} \dot{x} &= \mathcal{I} - g_1(x) - g_4(x) & \dot{x}_l &= f_l(x_l, \mathcal{I}) \\ \dot{y} &= g_1(x) - g_2(y) - g_5(y) & \dot{x}_\rho &= f_\rho(x_l, x_\rho) \\ \dot{z} &= g_2(y) - f(x)g_3(z) & \dot{x}_o &= f_o(x_l, x_\rho, x_o) \end{aligned}$$

- $g_j$  are kinetic functions
- $l = \text{input}; \rho = \text{regulatory}; o = \text{output}$

- Infinitesimal homeostasis** if and only if  $f_x = \frac{1}{g_3} \frac{g_1' g_2'}{g_2' + g_5'}$

Adapted from: [Reed, Best, Golubitsky, Stewart and Nijhout \(2018\)](#)





- [1] F. Antoneli, I. Stewart. *Symmetry and synchrony in coupled cell networks 1: Group networks*, International Journal of Bifurcation and Chaos, Vol. 16, No. 3 (2006), 559–577.
- [2] F. Antoneli, A.P.S. Dias, M. Golubitsky, Y. Wang. Synchrony in lattice differential equations. In: Some Topics In Industrial and Applied Mathematics. (R. Jeltsch, T. Li, and I. Sloan, eds.) Contemporary Applied Mathematics Series 8 World Scientific Publ. Co., 2007.
- [3] F. Antoneli, M. Golubitsky and I. Stewart. *Homeostasis in a feed forward loop gene regulatory network motif*. J. Theoretical Biology. 445 (2018) 103-109; DOI:10.1016/j.jtbi.2018.02.026
- [4] F. Antoneli, A.P.S. Dias, and R. Paiva. *Hopf Bifurcation in Coupled Cell Networks with Interior Symmetries*. SIAM Journal on Applied Dynamical Systems, Vol. 7, No. 1 (2008), 220-248.



- [5] H. Brandt. *Über eine Verallgemeinerung des Gruppenbegriffes*, Math. Ann. 96 (1927), 360–366.
- [6] R. Brown. *From groups to groupoids: A brief survey*, Bull. London Math. Soc. 19 (1987), 113–134.
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Grato pela atenção!