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Synchronyzation in Coupled Cell Networks: an algrebraic approach with pratical implementation

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February 6, 2026

Network Dynamics?

In many applications, ODEs are associated with a given network, and the form of these equations reflects the network architecture. For example in a neuroscience model, nodes might represent neurons and edges axons, coupled via electrical signals passing along the axons. The state of each node i is represented by a variable x_i, which might be a scalar or a vector.

Each node typically has an internal dynamic, an ODE that determines how it would behave if it were not coupled to other nodes. Connections from one node to another lead to coupling terms in the equations: if there is an input from node j to node i, then $\frac{dx_i}{dt}$ is a function of both x_i and x_j.

Network Dynamics

Motivation

Example (βIG model)

The βIG model of diabetes, Topp et al. (2000), takes the form

$$\begin{aligned} \dot{G} &= a - (b + cI)G \\ \dot{I} &= \beta\left(\frac{dG^2}{e + G^2}\right) - fI \\ \dot{\beta} &= (-g + hG - iG^2)/\beta \end{aligned} \tag{1}$$

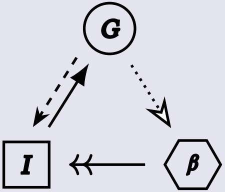
Here dots are time derivatives. The terms G = glucose level, I = insulin level, and β = beta-cell mass depend on time t. The other terms $a, b, c, d, e, f, g, h, i$ are parameters, whose value is constant during any particular run of the model or in any particular real system.

Network Dynamics

Motivation

Example (βIG model)

The network structure arises when we consider which variables depend on which.
Here:



- The change in G depends on G, I but not on β.
- The change in I depends on G, I, and β.
- The change in β depends on G, β but not on I.

It is natural to encode these relationships as the network. Here each variable is represented by a cell symbol (circle, square, hexagon) and arrows show which variable affects any given cell variable. The different cell symbols indicate different ‘cell types’, meaning that the form of the equation is different for those cells.

Example (FitzHugh-Nagumo neurons)

Consider an ODE system representing three coupled FitzHugh-Nagumo neurons:

$$\begin{aligned} \dot{v}_1 &= v_1(a - v_1)(v_1 - 1) - w_1 - cv_2 & \dot{w}_1 &= bv_1 - \gamma w_1 \\ \dot{v}_2 &= v_2(a - v_2)(v_2 - 1) - w_2 - cv_3 & \dot{w}_2 &= bv_2 - \gamma w_2 \\ \dot{v}_3 &= v_3(a - v_3)(v_3 - 1) - w_3 - cv_1 & \dot{w}_3 &= bv_3 - \gamma w_3 \end{aligned} \tag{2}$$

Here v_i is the membrane potential of cell i, w_i is a surrogate for an ionic current, and a; b; γ are parameters with $0 < a < 1$, $b > 0$; $\gamma > 0$.

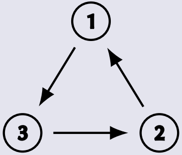
In (2) the dynamic equations are the same for each neuron, subject to appropriate permutations of the variables.

Network Dynamics

Motivation

Example (FitzHugh-Nagumo neurons)

In other words, the individual neurons are identical, and the couplings are also identical.



$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2), \\ \dot{x}_2 &= f(x_2, x_3), \\ \dot{x}_3 &= f(x_3, x_1), \end{aligned}$$

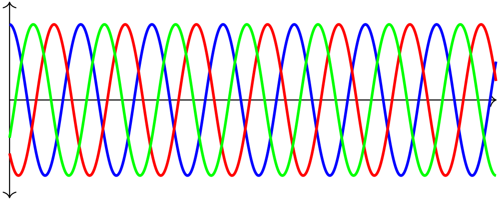
The state space of cell i is now 2-dimensional, with variables (v_i, w_i). Because the variables enter the equations in the same manner for each i, subject to the cyclic permutation $\mathbb{Z}_3$, the cells have the same type and so do the arrows. In the diagram, we represent this by using circles for all three cells and the same kind of arrow for all three couplings.

Network Dynamics

Motivation

We point out that when $a = b = \gamma = 0.5$ and $c = 2$, system (3) has a stable periodic state in which successive cells are one-third of a period out of phase. Below we show the pattern for v_j; the same pattern occurs for w_j. This state is a *discrete rotating wave* that exhibits spatiotemporal symmetry caused by the action of $\mathbb{Z}_3$:

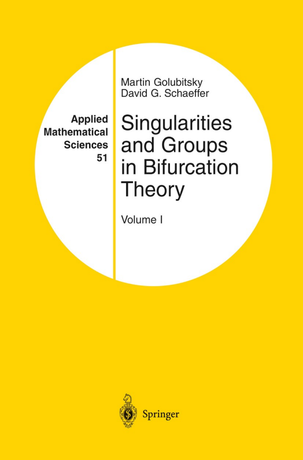
$$x_2(t) = x_1(t - T/3) \qquad x_3(t) = x_1(t - 2T/3)$$



This kind of dynamics (with the action of a symmetry group) is called Equivariant Dynamics!

Network Dynamics

Equivariant cases



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Applied Mathematical Sciences 51

Singularities and Groups in Bifurcation Theory

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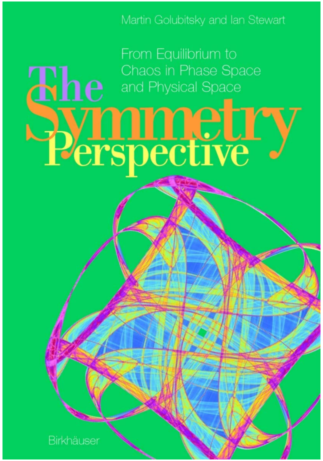


Martin Golubitsky and Ian Stewart

The Symmetry Perspective

From Equilibrium to Chaos in Phase Space and Physical Space

Birkhäuser

Network Dynamics

Historical notes

With appropriate adjustments, many results of equivariant dynamics apply to symmetric networks. However, few models in applied science exhibit global symmetries.

In 2002, Marcus Pivato described a 16-cell network that had a periodic state in which the nodes were partitioned into four subsets of four nodes. [15]

The cells in each partition were synchronous, while cells in different partitions exhibited the same dynamics within a phase shift multiple of 1/4 of the period. This was a rotating wave induced by $\mathbb{Z}_4$, except that the network **lacks** $\mathbb{Z}_4$ symmetry.

Network Dynamics

Historical notes

Around 2003, Marty Golubitsky, Ian Stewart, and other collaborators proposed a theory to study this type of network based on the notion of local symmetry, establishing the **groupoid formalism**. [10, 12]

Mathematically, a network is represented by a directed graph whose nodes and edges are classified according to associated labels or “types”.
The nodes (or “cells”) of a (directed and labeled) network $\mathcal{N}$ represent dynamical systems (state variables), and the edges (“arrows”) represent couplings, interactions between these variables.

Formally, we have:

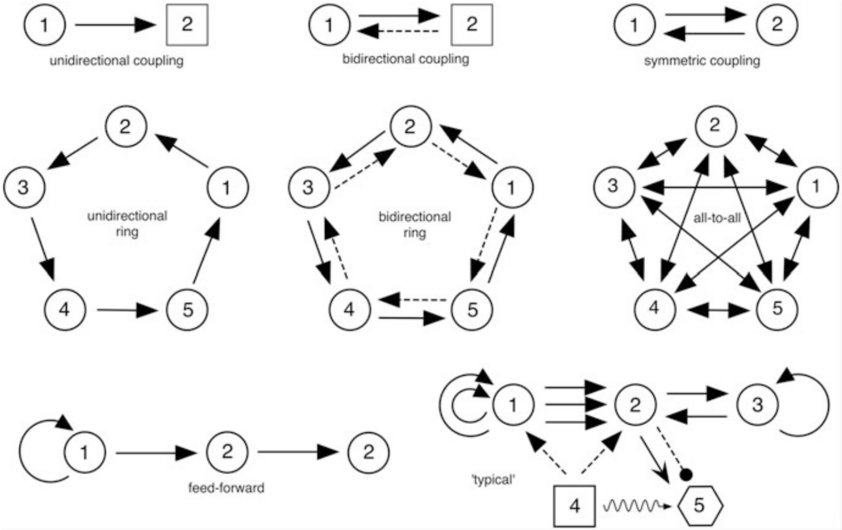
Coupled Networks

Coupled Network Definition

A coupled cell network $\mathcal{N} = \mathcal{N}(\mathcal{C}, \mathcal{E})$ satisfies the following conditions:

1. There is a finite set $\mathcal{C} = \{1, 2, \ldots, n\}$ of cells;
2. There is a finite set $\mathcal{E}$ of arrows.
3. Each arrow e has a head cell $\mathcal{H}(e) \in \mathcal{C}$ and a tail cell $\mathcal{T}(e) \in \mathcal{C}$.
4. Cells are classified into types. Formally, this is done by defining an equivalence relation $\sim_C$ on $\mathcal{C}$, called cell equivalence. Cells are equivalent if they have the same type.
5. Arrows are also classified into types by defining an equivalence relation $\sim_E$ on $\mathcal{E}$, called arrow equivalence. Arrows are equivalent if they have the same type.
6. Types satisfy two compatibility conditions. If $e_1, e_2 \in \mathcal{E}$ are $\sim_E$-equivalent, then $\mathcal{H}(e_1)$ e $\mathcal{H}(e_2)$ are $\sim_C$-equivalent, and similarly, $\mathcal{T}(e_1)$ e $\mathcal{T}(e_2)$.

Coupled Network Dynamics



Coupled Network Dynamics

Definition of Input Set

Let $c \in \mathcal{C}$. The *input set* of c is the set $I(c)$ of all arrows $e \in \mathcal{E}$ such that $\mathcal{H}(e) = c$. That is,

$$I(c) = \{e \in \mathcal{E} \mid \mathcal{H}(e) = c\}$$

Definition of Input Isomorphism

An *input isomorphism* $\beta : I(c) \longrightarrow I(d)$ is a bijection between input sets that preserves the arrow type, that is, $e \sim_E \beta(e)$ for all β and every arrow $e \in I(c)$.

(Note that $\beta^{-1}(e')$ is $\sim_E$-equivalent to e' for all $e' \in I(d)$)

Coupled Network Dynamics

If there exists an input isomorphism $\beta : I(c) \longrightarrow I(d)$, we say that c, d are *input-isomorphic* or **input-equivalent**.

Definition of $\sim_E$-equivalence

The input-equivalence relation $\sim_E$ in $\mathcal{C}$ is defined by $c \sim_E d$ if and only if there exists a bijection

$$\beta : I(c) \longrightarrow I(d)$$

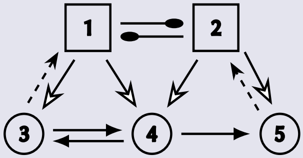
such that, for each $i \in I(c)$,

$$i \sim_E \beta(i)$$

Coupled Network Dynamics

Example (input isomorphism)

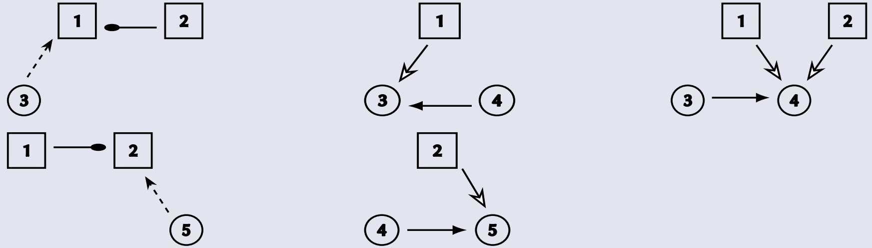
Let $\mathcal{N}$ be the network shown below:



We see that cells 1 and 2 are *input isomorphic*, as are cells 3 and 5. However, cells 1 and 3 are not input isomorphic. Although they both receive two inputs, the arrow types are different.

Coupled Network Dynamics

Example (input sets)



From left to right: $I(1), I(3), I(4), I(2), I(5)$.

Strictly, the arrows constitute the input set, but is convenient to show the head and tail cells as well.

Synchrony and Colorings

Synchrony in Symmetric Networks

We will now see how synchrony manifests itself in networks.

A **polydiagonal** is a subspace

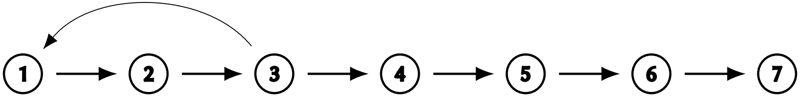
$$\Delta = \{x \mid x_c = x_d \text{ for some subset of cells}\}$$

A **synchrony subspace** is a flow-invariant polydiagonal.

Synchrony and Colorings

Synchrony in Generic Networks

Consider the following feed-forward network with its admissible functions:



$$\dot{x}_1 = f(x_1, x_3), \qquad \dot{x}_2 = f(x_2, x_1), \qquad \dot{x}_3 = f(x_3, x_2),$$

$$\dot{x}_4 = f(x_4, x_3), \qquad \dot{x}_5 = f(x_5, x_4), \qquad \dot{x}_7 = f(x_7, x_6),$$

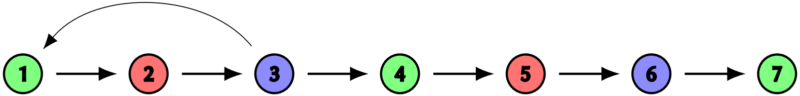
$$\dot{x}_7 = f(x_7, x_6),$$

We can split this network using **colorings** where $\Delta = \{x \mid x_1 = x_4 = x_7; x_2 = x_5; x_3 = x_6\}$ is flow-invariant.

Synchrony and Colorings

Synchrony in Generic Networks

Consider the following feed-forward network with its admissible functions:



$$
\begin{aligned}
\dot{x}_1 &= f(x_1, x_3), & \dot{x}_2 &= f(x_2, x_1), & \dot{x}_3 &= f(x_3, x_2), \\
\dot{x}_4 &= f(x_4, x_3), & \dot{x}_5 &= f(x_5, x_4), & \dot{x}_7 &= f(x_7, x_6), \\
\dot{x}_7 &= f(x_7, x_6), & & & &
\end{aligned}
$$

We can split this network using **colorings** where $\Delta = \{x \mid x_1 = x_4 = x_7; x_2 = x_5; x_3 = x_6\}$ is flow-invariant.

Synchronization and Colorings

Definition (Coloring)

A *coloring* of a network $\mathcal{G}$ is a map

$$\kappa : \mathcal{C} \longrightarrow K$$

where K is a set whose elements are called *colors*.

We say that cells c and d *have the same color* if $\kappa(c) = \kappa(d)$ and we write $c \sim_{\kappa} d$ (color equivalence).

A coloring κ is *balanced* if, whenever c and d have the same color, then there exists an input isomorphism $\beta : I(c) \longrightarrow I(d)$ such that i and $\beta(i)$ have the same color for all $i \in \mathcal{T}(I(i))$.

Synchronization and Colorings

In practice, a coloring is balanced if there exists an input isomorphism that preserves the colors of any two cells of the same color. Since cells of the same color must be input equivalent, a balanced coloring is a refinement of input equivalence.

The *polydiagonal* defined by a coloring κ of $\mathcal{G}$ is the space

$$\Delta_\kappa = \{x \mid \kappa(c) = \kappa(d) \implies x_c = x_d\}$$

That is, cells of the same color are synchronous for $x \in \Delta_\kappa$.

Theorem ([12], Theorem 4.3)

A polydiagonal Δ_κ is invariant for every admissible function if and only if κ is balanced.

Calculating Colorings

The set of all balanced equivalence relations has a partially ordered structure, using the relation of refinement. Let $\bowtie_i$ and $\bowtie_j$ be balanced equivalence relations on the set $\mathcal{C}$. Recall that $\bowtie_i$ refines $\bowtie_j$, denoted by $\bowtie_i \prec \bowtie_j$, if and only if $c \bowtie_i d \Rightarrow c \bowtie_j d$. where $c, d \in \mathcal{C}$. The set of all balanced equivalence relations of a (locally finite) network form a complete lattice in general.

Definition

Let $B = (b_{ij})$ be a $p \times q$ symbolic matrix. We say B is a *homogenous block matrix* if the sum $\sum_{j=1}^{q} b_{ij}$ is identical for all rows $i = 1, \ldots, p$.

Theorem

A polydiagonal subspace $\Delta_{\bowtie}$ is a synchrony subspace if and only if each block of the adjacency matrix A, which corresponds to a block of $P_{\bowtie}$, is a homogeneous block matrix.

Calculating Colorings

Alternatively, the last theorem can be obtained by defining one (integer entry) adjacency matrix per arrow type, and finding the intersection of balanced equivalence relations for the arrow-specific adjacency matrices.

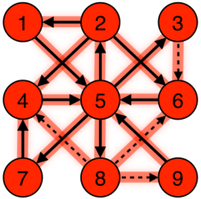
Following this idea leads to a computer algorithm which determines all balanced equivalence relations.

We enumerate all possible equivalence relations $\bowtie$ of n-cells, and test which are balanced. If the top lattice node is known in advance, e.g. using the algorithm in Aldis [1], only equivalence relations which refine this top node need be tested. To test if each $\bowtie$ is balanced, we construct an $n \times k$ matrix, where k is the number of equivalence classes of $\bowtie$, from the $n \times n$ adjacency matrix of $\mathcal{G}$.

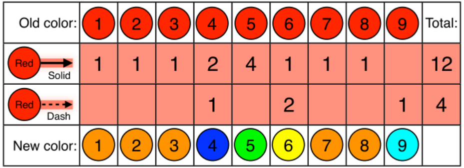
Calculating Colorings

Step 0: Start by assigning the same color (node class) to each node, here shown in red. If multiple node types are considered as in Aldis, then each node type would be allocated a unique color.

Step 1: In each step compute the "input driven refinement" by tallying the inputs to each node according to the color of the node the input is from, and the arrow type. After tabulation, unique input combinations give the next node partition. See the figure below. Here we have one node color (red), and two arrow types (solid and dashed), so for each node there are two input counts (solid from red, dashed from red).



Old partition: (123456789)



Old color:	1	2	3	4	5	6	7	8	9	Total:
Red Solid	1	1	1	2	4	1	1	1		12
Red Dash				1		2			1	4
New color:	1	2	3	4	5	6	7	8	9	

New partition: (12378)(4)(5)(6)(9)

Calculating Colorings

We observe five unique input combinations, and so assign them five colors as the "input driven refinement" of the (trivial) input partition. For instance, nodes with one solid input from a red node only have been assigned the new partition color orange.

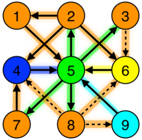
Step 2: There are now five node colors (shown here as orange, blue, green, yellow and cyan). Thus with two arrow types (solid and dashed), we consider ten input types ($5 \times 2 = 10$; solid from orange,..., solid from cyan, dashed from orange,..., dashed from cyan).

Step 3: There are now six node colors (pink, purple, blue, green, yellow and cyan), so with two arrow types (solid and dashed) we consider twelve input types ($6 \times 2 = 12$; solid from pink,..., solid from cyan, dashed from pink,..., dashed from cyan).

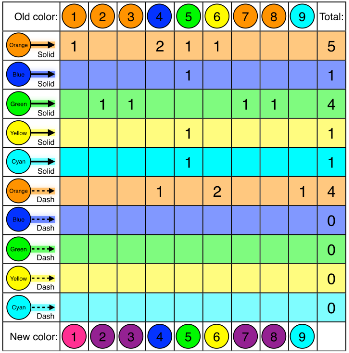
At this iteration the partition of nodes is unchanged, and the algorithm halts. This gives the top lattice node.

Calculating Colorings

Step 2



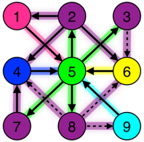
Old partition: (12378)(4)(5)(6)(9)



Old color:	1	2	3	4	5	6	7	8	9	Total:
Orange Solid	1			2	1	1				5
Blue Solid					1					1
Green Solid		1	1				1	1		4
Yellow Solid					1					1
Cyan Solid					1					1
Orange Dash				1		2			1	4
Blue Dash										0
Green Dash										0
Yellow Dash										0
Cyan Dash										0
New color:	1	2	3	4	5	6	7	8	9	

New partition: (1)(2378)(4)(5)(6)(9)

Step 3



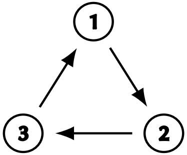
Old partition: (1)(2378)(4)(5)(6)(9)

Old color:	1	2	3	4	5	6	7	8	9	Total:
Pink Solid					1					1
Purple Solid	1			2		1				4
Blue Solid					1					1
Green Solid		1	1				1	1		4
Yellow Solid					1					1
Cyan Solid					1					1
Pink Dash										0
Purple Dash				1		2			1	4
Blue Dash										0
Green Dash										0
Yellow Dash										0
Cyan Dash										0
New color:	1	2	3	4	5	6	7	8	9	

New partition: (1)(2378)(4)(5)(6)(9)

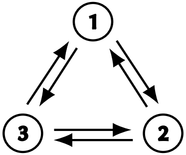
Resuming

$$\begin{aligned}\dot{x}_1 &= f(x_1, x_3),\\ \dot{x}_2 &= f(x_2, x_1),\\ \dot{x}_3 &= f(x_3, x_2),\end{aligned}$$



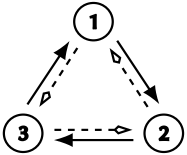
$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned}\dot{x}_1 &= g(x_1, \overline{x_2, x_2}),\\ \dot{x}_2 &= g(x_2, \overline{x_3, x_1}),\\ \dot{x}_3 &= g(x_3, \overline{x_1, x_2}),\end{aligned}$$



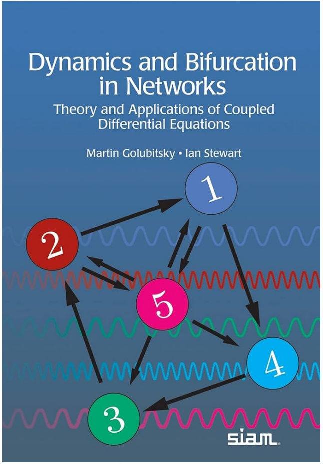
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned}\dot{x}_1 &= h(x_1, x_2, x_2),\\ \dot{x}_2 &= h(x_2, x_3, x_1),\\ \dot{x}_3 &= h(x_3, x_1, x_2).\end{aligned}$$



$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

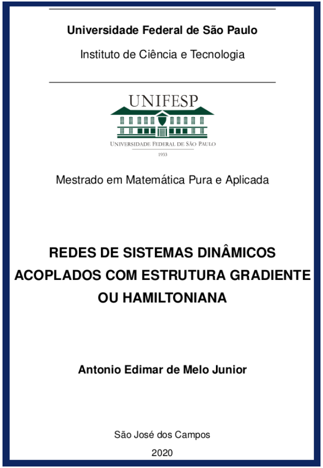
Further reading



Dynamics and Bifurcation in Networks

Theory and Applications of Coupled Differential Equations

Martin Golubitsky • Ian Stewart



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Mestrado em Matemática Pura e Aplicada

REDES DE SISTEMAS DINÂMICOS ACOPLADOS COM ESTRUTURA GRADIENTE OU HAMILTONIANA

Antonio Edimar de Melo Junior

São José dos Campos

2020

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